From (1.6) we derive formulas for the radius of the contact circle a and the approach of the bodies δ

$$a = [3\,(8\pi)^{-1} RPm_0]^{1/3}, \quad \delta = [3\,(8\pi)^{-1} PR^{1/2} m_0]^{2/3} \qquad (2.19)$$

According to (1.4), the elastic displacements of points of the bodies on the contact section are determined by the formulas

$$w_j = m_j^0 m_0^{-1}\,(\delta - \rho^2 / 2R), \quad \rho^2 = x^2 + y^2 \qquad (2.20)$$

The relationships (2.19), (2.20) differ from the corresponding Hertz formulas for an isotropic medium just by the value of the constants m_0, m_j^0.

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CONTACT PROBLEM FOR A PLANE CONTAINING A SLIT OF VARIABLE WIDTH

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L. T. BOIKO and P. E. BERKOVICH
(Dnepropetrovsk)

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The problem of compression of an elastic plane with a slit of variable width commensurate to the elastic strains is considered. The case of the origination of several contact sections of the slit edges is investigated. Adhesion of the edges hence occurs at some part of the contact area, while slip is possible at the rest of this area. A solution of the problem is obtained in quadratures by the Muskhelishvili method using the apparatus of linear conjugates of analytic functions. The stress and displacement potentials are found, the magnitudes of the contact sections and the adhesion zones are determined. A specific example is analyzed and numerical computations are carried out.

The contact problem for a plane weakened by a constant-width rectilinear slit has been considered in [1 – 3].

1. An infinite elastic isotropic plane is weakened by a variable width rectilinear slit $h(x)$ commensureate with the elastic strains. The plane is compressed by uniformly distributed stress resultants with components P and T (Fig. 1), applied at infinity. The slit edges make contact along the sections (α_k, β_k) during deformation. Each contact area consists of an adhesion section of the edges (c_k, d_k) and two sections (α_k, c_k) and (d_k, β_k) on which slip is possible.

Let us use the notation: L_1 is the set of adhesion sections, L_2 is the set of slip sections,

L_3 is the set of free sections, $\rho\,(x)$ is the friction coefficient (we take the friction law in the Coulomb form). The remaining notation agrees with [4]. The boundary conditions of the problem are

$$Y_y^+ = Y_y^-, \qquad X_y^+ = X_y^- \quad \text{on } L = L_1 + L_2 + L_3 \tag{1.1}$$

$$v'^+ - v'^- = -h'(x) \quad \text{on } L_1 + L_2, \quad Y_y^+ = 0 \text{ on } L_3 \tag{1.2}$$

$$u'^+ - u'^- = 0 \quad \text{on } L_1, \quad X_y^+ = \rho Y_y^+ \quad \text{on } L_2, \quad X_y^+ = 0 \quad \text{on } L_3 \tag{1.3}$$

We express the stress and displacement tensor components in terms of two piecewise analytic functions of a complex variable $\Phi\,(z)$ and $\Omega\,(z)$ [4]

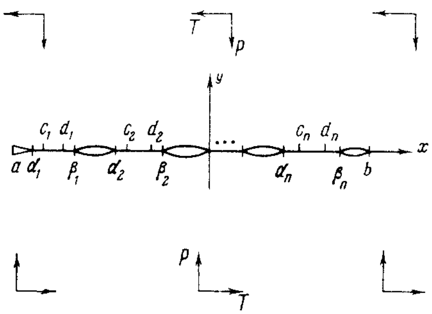

Fig. 1

$$Y_y - iX_y = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\,\overline{\Phi'(z)}$$

$$2\mu(u' + iv') = \varkappa\Phi(z) - \Omega(\bar{z}) - (z - \bar{z})\,\overline{\Phi'(z)}$$

The stresses far from the slit are bounded and the principal vector of the external stress resultants applied to the slit contour is zero, hence

$$\Phi(z) = \Gamma + O(z^{-2}), \quad \Omega(z) = \Gamma + \bar{\Gamma} + O(z^{-2}), \quad |z| \gg b - a \tag{1.4}$$

$$\Gamma = -\frac{1}{4}\sqrt{P^2 + T^2}, \quad \Gamma' = B' + iC' = \frac{T^2 - P^2}{2\sqrt{T^2 + P^2}} - i\,\frac{TP}{\sqrt{T^2 + P^2}}$$

From the boundary condition (1.1) we obtain

$$[\Phi - \Omega]^+ = [\Phi - \Omega]^- \quad \text{on } L$$

from which there follows

$$\Phi(z) = \Omega(z) - \bar{\Gamma}' \tag{1.5}$$

Using (1.5), let us express the stresses and displacements in terms of the function $\Omega\,(z)$. On the slit contour

$$2Y_y^{\pm} = [\Omega + \bar{\Omega}]^+ + [\Omega + \bar{\Omega}]^- - 2B' \tag{1.6}$$

$$2iX_y^{\pm} = [-\Omega + \bar{\Omega}]^+ + [-\Omega + \bar{\Omega}]^- - 2iC'$$

$$4\mu u'^{\pm} = [\varkappa\Omega - \bar{\Omega}]^{\pm} - [\Omega - \varkappa\bar{\Omega}]^{\mp} - 2\varkappa B' \tag{1.7}$$

$$4\mu i v'^{\pm} = [\varkappa\Omega + \bar{\Omega}]^{\pm} - [\Omega + \varkappa\bar{\Omega}]^{\mp} + 2i\varkappa C'$$

Substituting (1.6) and (1.7) into the boundary condition (1.2), we arrive at the problem of a linear conjugate with discontinuous coefficients

$$[\Omega + \bar{\Omega}]^+ - [\Omega + \bar{\Omega}]^- = -4\pi i\gamma h'(x) \quad \text{on } L_1 + L_2$$

$$[\Omega + \bar{\Omega}]^+ + [\Omega + \bar{\Omega}]^- = 2B' \quad \text{on } L_3, \quad \left(\gamma = \frac{\mu}{\pi(\varkappa + 1)}\right)$$

The solution of this problem is written as follows

$$\Omega(z) + \bar{\Omega}(z) = -2\gamma X_1(z) \int\limits_{L_1 + L_2} \frac{h'(t)\,dt}{X_1^+(t)\,(t - z)} + B' + [D_0 - B'G(n)]\,X_1(z) \tag{1.8}$$

Here

$$X_1(z) = \prod_{k=1}^{n} \sqrt{(z-\alpha_k)(z-\beta_k)} \,/\, \sqrt{(z-a)(z-b)}, \quad G(n) = \begin{cases} 1, & n=1 \\ 0, & n \geqslant 2 \end{cases}$$

By virtue of (1.4), the constants D_0, α_k and β_k satisfy the following $(n+1)$ equations

$$D_0 = \begin{cases} 2\Gamma + 2B', & n=1 \\ 0, & n \geqslant 2 \end{cases} \tag{1.9}$$

$$\int\limits_{L_1+L_2} \frac{h'(t)\, t^k dt}{X_1^+(t)} = \frac{2\Gamma + B'}{2\gamma} A_k, \quad k = 0, 1, \ldots, n-1$$

Hence

if $n = 1$, then $A_0 = -{}^1/_2 (a + b - \alpha_1 - \beta_1)$

if $n = 2$, then $A_1 = -\frac{1}{2}\left[a + b - \sum_1^2 (\alpha_j + \beta_j)\right]$, $A_0 = 1$

if $n \geqslant 3$, then

$$A_{n-1} = -\frac{1}{2}\left[a + b - \sum_1^n (\alpha_j + \beta_j)\right], \; A_{n-2} = 1, \; A_k = 0, \; k \leqslant n-3$$

The missing n equations to determine the coordinates of the ends of the contact sections are obtained from the conditions

$$v^+(\alpha_k) - v^-(\alpha_k) = -h(\alpha_k), \quad k = 1, 2, \ldots, h \tag{1.10}$$

From (1.7) we have

$$v^+(x) - v^-(x) = \frac{1}{4\gamma\pi i} \int_a^x Y(t)\, dt, \quad Y(t) = [\Omega + \overline{\Omega}]^+ - [\Omega + \overline{\Omega}]^- \tag{1.11}$$

Substituting (1.11) into (1.10), we find the desired equations

$$\int_a^{\alpha_1} Y(t)\, dt = -4\pi i \gamma h(\alpha_1) \tag{1.12}$$

$$\int_{\beta_k}^{\alpha_{k+1}} Y(t)\, dt = -4\pi i \gamma [h(\alpha_{k+1}) - h(\beta_k)], \quad k = 1, 2, \ldots, n-1$$

Thus, we have a complete system of equations to determine the contact sections.

By using (1.6) – (1.8), we obtain the following linear conjugate problem from the last boundary condition in (1.3):

$$\Omega^+ - \Omega^- = -2\pi i \gamma h'(x) \quad \text{on } L_1$$

$$\Omega^+ + \Omega^- = [1 - i\rho(x)]\, g(x) + \overline{\Gamma}' \quad \text{on } L_2$$

$$\Omega^+ + \Omega^- = \overline{\Gamma}' \quad \text{on } L_3$$

$$g(x) = -2\gamma X_1^+(x) \int\limits_{L_1+L_2} \frac{h'(t)\, dt}{X_1^+(t)\,(t-x)} + [D_0 - B'G(n)]\, X_1^+(x)$$

Solving this problem, we determine the function $\Omega(z)$

$$\Omega(z) = \frac{X_2(z)}{2\pi i}\left\{-2\pi i\gamma \int_{L_1} \frac{h'(t)\,dt}{X_2^+(t)(t-z)} + \int_{L_2} \frac{[1-i\rho(t)]\,g(t)\,dt}{X_2^+(t)(t-z)}\right\} + \tag{1.13}$$

$$\frac{1}{2}\bar{\Gamma}' + \left[C_0 - \frac{1}{2}G(m)\bar{\Gamma}'\right]X_2(z)$$

$$G(m) = \begin{cases} 1, & m=1 \\ 0, & m \geqslant 2 \end{cases}, \quad X_2(z) = \prod_{k=1}^{m} \sqrt{(z-c_k)(z-d_k)} \,/\, \sqrt{(z-a)(z-b)}$$

(m is the number of adhesion sections). By virtue of (1.4), the constants C_0, c_k and d_k satisfy the following $(m+1)$ equations:

$$C_0 = \begin{cases} \Gamma + \bar{\Gamma}', & m=1 \\ 0, & m \geqslant 2 \end{cases} \tag{1.14}$$

$$\gamma \int_{L_1} \frac{h'(t)\,t^k dt}{X_2^+(t)} - \frac{1}{2\pi i}\int_{L_2} \frac{[1-i\rho(t)]\,g(t)\,t^k dt}{X_2^+(t)} = \left(\Gamma + \frac{1}{2}\bar{\Gamma}'\right) B_k$$

$$k = 0, 1, \ldots, m-1$$

Hence

if $m = 1$, then $B_0 = -{}^1/_2\,(a + b - c_1 - d_1)$

if $m = 2$, then $B_1 = -\frac{1}{2}\left[a + b - \sum_1^2 (c_j + d_j)\right]$, $B_0 = 1$

if $m \geqslant 3$, then

$$B_{m-1} = -\frac{1}{2}\left[a + b - \sum_1^m (c_j + d_j)\right], \quad B_{m-2} = 1, \quad B_k = 0, \quad k \leqslant m-3$$

We obtain the missing m equations to determine the coordinates of the ends of the adhesion sections from the conditions

$$u^+(c_k) - u^-(c_k) = \int_a^{c_k} (u'^+ - u'^-)\,dt = 0, \quad k = 1, 2, \ldots, m$$

Using (1.7) we find

$$\int_a^{c_1} [\Omega^+ - \Omega^-]\,dt = -2\pi i\gamma h(c_1) \tag{1.15}$$

$$\int_{d_k}^{c_{k+1}} [\Omega^+ - \Omega^-]\,dt = -2\pi i\gamma\,[h(c_{k+1}) - h(d_k)], \quad k = 1, 2, \ldots, m-1$$

Thus, we have a complete system of equations for determination of the adhesion sections. We find the contact stresses from (1.6)

$$Y_y^+ = g(x) \quad \text{on } L_1 + L_2,$$

$$X_y^+ = i\,[2C_0 - G(m)\bar{\Gamma}']\,X_2^+(x) - ig(x) - 2\gamma i X_2^+(x) \int_{L_1} \frac{h'(t)\,dt}{X_2^+(t)(t-x)} +$$

$$\frac{X_2^+(x)}{\pi} \int_{L_2} \frac{[1-i\rho(t)]\,g(t)\,dt}{X_2^+(t)(t-x)} \quad \text{on } L_1$$

$$X_y^+ = \rho(x)\,Y_y^+ \quad \text{on } L_2$$

2. As an illustration, let us consider the contact problem for a slit whose width varies according to a linear law

$$h(x) = h_2 + (h_1 - h_2)\frac{b-x}{b-a}, \quad h'(x) = -\frac{h_1 - h_2}{b-a}$$

We consider the friction coefficient constant $\rho(x) = \rho_0$. In this case one contact section (α, β) originates during deformation, therefore

$$X_1(z) = \sqrt{\frac{(z-\alpha)(z-\beta)}{(z-a)(z-b)}}, \quad X_2(z) = \sqrt{\frac{(z-c)(z-d)}{(z-a)(z-b)}}$$

We use the notation

$$D_1 = \frac{P^2}{\sqrt{P^2+T^2}}, \quad C_1 = \frac{P(P\rho_0 - T)}{\sqrt{P^2+T^2}}$$

The location of the contact area is determined from (1.9) and (1.12)

$$4\gamma h'\int_\alpha^\beta \frac{dt}{X_1^+(t)} - D_1(a+b-\alpha-\beta) = 0 \tag{2.1}$$

$$2i\gamma h'\int_a^\alpha X_1^+(x)\int_\alpha^\beta \frac{dt}{X_1^+(t)(t-x)}dx + D_1 i\int_a^\alpha X_1^+(x)\,dx + 2\pi\gamma h(\alpha) = 0$$

At the time of the beginning of contact $\alpha = \beta$. Passing to the limit as $\alpha \to \beta \to \alpha_*$ in the system (2.1), we find the point at which contact originates, and the appropriate load at this instant

$$\alpha_* = \frac{h_1 b + h_2 a}{h_1 + h_2}, \quad D_1{}^* = 4\gamma\pi\frac{\sqrt{h_1 h_2}}{b-a}$$

From (1.13) we find an expression for the function $\Omega(z)$

$$\Omega(z) = \frac{1}{2}\bar{\Gamma}' - \frac{1}{2}(1 - i\rho_0)D_1 X_1(z) - \frac{i}{2}[D_1\rho_0 + C']X_2(z) -$$

$$\gamma h'\left[(1-i\rho_0)X_1(z)\int_\alpha^\beta \frac{dt}{X_1^+(t)(t-z)} + i\rho_0 X_2(z)\int_c^d \frac{dt}{X_2^+(t)(t-z)}\right]$$

From (1.14) and (1.15) we have a system of equations to determine the coordinates of the ends of the adhesion sections

$$4\gamma h'\int_c^d \frac{dt}{X_2^+(t)} - C_1(a+b-c-d) = 0$$

$$2i\gamma h'\int_a^c X_2^+(x)\int_c^d \frac{dt}{X_2^+(t)(t-x)}dx + C_1\int_a^c X_2^+(x)\,dx + 2\gamma h(c) = 0$$

Passing to the limit as $c \to d \to c_*$, we find the load and the point of origination of the adhesion section

$$C_1{}^* = 4\gamma\pi\frac{\sqrt{h_1 h_2}}{b-a}\rho_0, \quad c_* = \frac{h_1 b + h_2 a}{h_1 + h_2}$$

As is seen from these latter formulas, the adhesion section originates at the point in which contact of the slit edges starts. In this case the stresses on the contact area are determined by the formulas

$$Y_y{}^+ = -X_1{}^+(x)\left\{2\gamma h'\int_\alpha^\beta \frac{dt}{X_1^+(t)(t-x)} + D_1\right\}$$

$$X_y{}^+ = -X_2{}^+(x)\left\{2\gamma h'\rho_0\int_c^\alpha \frac{dt}{X_2^+(t)(t-x)} + C_1\right\} + \rho_0 Y_y{}^+ \quad \text{on } L_1$$

$$X_y{}^+ = \rho_0 Y_y{}^+ \quad \text{on } L_2$$

Presented in Fig. 2 are graphs of the contact stresses for the case $h_1 = 0.0002\ (b - a)$, $h_2 = 0.1\ h_1$, $(x_* = x\ /\ (b - a))$. The method of chords [5, 6] is used for numerical solution of the system of transcendental equations. The numbers on the curves in Fig. 2 correspond to the values $10^4 \cdot P\ /\ E$ (E is the elastic modulus of the material). For all the curves $T = 0.1\ P$. As is seen from the graphs, the ends of the adhesion section are not singularities for the normal stresses.

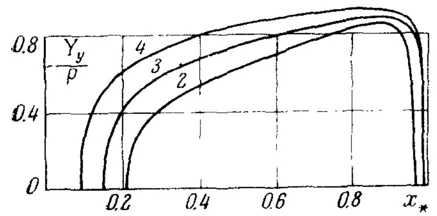

Fig. 2

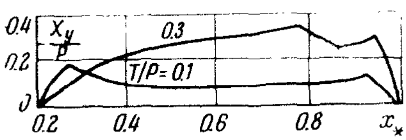

Fig. 3

For both curves $P = 2 \cdot 10^{-4}\ E$ in Fig. 3. As is seen from the graphs, the shear stress is continuous at the points separating the adhesion and slip sections, but its derivative undergoes a discontinuity. Diminution of the angle between the slit line and the line of load action, i.e. increasing the component T, results in diminution of the adhesion zone.

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